NECESSARY AND SUFFICIENT CONDITION FOR THE BOUNDEDNESS OF SOLUTIONS OF A CLASS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

(NEOBKHODIMOE I DOSTATOCHNOE USLOVIE OGRANICHENNOSTI RESHENII ODNOGO KLASSA SISTEM LINEINYKH DIFFERENTSIAL'NYKH URAVNENII)

PMM Vol.28, № 1, 1964, pp.149-150

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(Received April 11, 1963)

In investigating the stability of solutions of nonlinear systems of differential equations by the comparison method*, the question of boundedness of solutions of systems of the form

$$x_i' = a_i(t)\, x_i + \sum_{i\neq j} b_{ij}(t)\, x_j \qquad \begin{pmatrix} i = 1, \ldots, n \\ j = 1, \ldots, n \end{pmatrix} (b_{ij} \geqslant 0) \tag{1}$$

arises

For the more general class of systems

$$x' = (A + B)\, x \tag{2}$$

where $A = \{a_{ij}(t)\}$ and $B = \{b_{ij}(t)\}$ are square $(n \times n)$ matrices, it is known [1] that if the solutions of the system $y' = Ay$ are bounded and

$$\int^{t} a_{ii}(s)\, ds \geqslant \alpha > -\infty \qquad (i = 1, \ldots, n)$$

then for

$$\int_{t_0}^{\infty} \|B(s)\|\, ds < \infty \tag{3}$$

the solutions of the system (2) are also bounded. The theorem presented below shows that condition (3) is necessary if the system (2) has the form (1).

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* L.F. Rakhmatullina, Primenenie integral'nykh neravenstv k issledovaniiu ustoichivosti reshenii differentsial'nykh uravnenii (Application of integral inequalities to the investigation of the stability of solutions of differential equations). Dissertation, Kazan' University, 1963.

Theorem. If

$$\int_{t_0}^{t} a_i(s)\,ds \geqslant \alpha > -\infty \qquad (i = 1, \ldots, n)$$

then the solutions of the system (1) are bounded if, and only if

$$\varphi_i(t) = \int_{t_0}^{t} a_i(s)\,ds \leqslant \beta < \infty, \quad \psi_{ij}(t) = \int_{t_0}^{\infty} b_{ij}(s)\,ds < \infty \qquad (i, j = 1, \ldots, n;\ i \neq j)$$

For the proof let us consider the system of Volterra equations

$$y_i(t) = \int_{t_0}^{t} \sum_{j=1,\, j\neq i}^{n} K_{ij}(t, s)\, y_j(s)\,ds + f_i(t) \quad \left(K_{ij}(t, s) = b_{ij}(s) \exp \int_{s}^{t} a_i(\tau)\, d\tau\right) \tag{4}$$

If

$$f_i(t) = x_i(t_0) \exp \varphi_i(t)$$

then the systems (1) and (4) are equivalent. If the above conditions are satisfied then

$$\lim_{T\to\infty} \overline{\lim_{t\to\infty}} \int_{T}^{t} K_{ij}(t, s)\,ds \leqslant \lim_{T\to\infty} e^{\beta-\alpha} \int_{T}^{\infty} b_{ij}(s)\,ds = 0$$

From this and from Theorem 1 of [3] the boundedness of the solutions of the system (4) follows for bounded $f_i(t)$. By virtue of the mentioned equivalence and boundedness of the functions

$$f_i(t) = x_i(t_0) \exp \varphi_i(t)$$

we obtain the boundedness of the solutions of the system (1).

If $x_i(t_0) \geqslant 0$, then by virtue of Wazewski's theorem on differential inequalities [3](or the theorem on an integral inequality [4]), we have $x_i(t) \geqslant 0$ for $t \geqslant t_0$. Hence, it follows from (4) that $x_i(t) \geqslant x_i(t_0) \exp \varphi_i(t)$, since $K_{ij}(t, s) \geqslant 0$. Therefore,

$$\beta_i = \sup_t \varphi_i(t) < \infty \tag{5}$$

if the $x_i(t)$ are bounded.

For bounded $f_i(t)$ solutions of the system (4) are bounded if the solutions of the system (1) are bounded. In fact let $|f_i(t)| < \gamma$ and $\{x_1(t), \ldots, x_n(t)\}$ be such a solution of (1) that $x_i(t_0) = \gamma e^{-\alpha}$. Since

$$|y_i(t)| \leqslant \int_{t_0}^{t} \sum_{j\neq i} K_{ij}(t, s)\, |y_j(s)|\,ds + \gamma \exp\{\varphi_j(t) - \alpha\}$$

$$x_i(t) = \int_{t_0}^{t} \sum_{j\neq i} K_{ij}(t, s)\, x_j(s)\,ds + \gamma \exp\{\varphi_i(t) - \alpha\}$$

then by virtue of the theorem on an integral inequality [4]

$$|y_i(t)| = x_i(t) \qquad \text{for } t \geqslant t_0$$

From the boundedness of the solutions of the system (4) for bounded $f_i(t)$ and from Lemma 1 of [3] it follows that

$$\sup_t \int_{t_0}^{t} K_{ij}(t, s)\,ds < \infty$$

Hence and from (5) we have

$$\int_{t_0}^{t} b_{ij}(s)\,ds \leqslant e^{\beta_i - \alpha} \sup_t \int_{t_0}^{t} K_{ij}(t, s)\,ds < \infty$$

The theorem is proved.

In discussing the elucidated theorem at an Izhevsk seminar, V.I. Logunov indicated the following statement of [6]: solutions of the system $x_1' = ax_2$, $x_2' = bx_1 + cx_2$ with non-negative coefficients are bounded if, and only if

$$\int_{t_0}^{\infty} b(s)\,ds < \infty, \qquad \int_{t_1}^{\infty} c(s)\,ds < \infty, \qquad \int_{t_0}^{\infty} a(s) \int_{t_0}^{s} b(t)\,dt\;ds < \infty$$

It is easy to see that the system

$$x_1' = \exp\{t^{-1}\}\;x_2, \qquad x_2' = t^{-2}\,x_2\;(t_0 = 1)$$

satisfies the presented conditions but has the unbounded solution

$$x_1 = t,\; x_2 = \exp\{t^{-1}\}.$$

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Translator's note. The references appear to have been confused in the original text, a sixth reference apparently having been ommitted in the bibliography. Those references which were obviously incorrect have been changed. Reference [6] in the text has been retained, although it is not mentioned in the bibliography.

Translated by M.D.F.